Math Virtual Learning

Precalculus with Trigonometry

May 8, 2020

Precalculus with Trigonometry
Lesson: May 8th, 2020

Objective/Learning Target:
Students will solve Trigonometric Equations using identities.

Before we start, review the following identities.

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

Reciprocal Identities

$\sin\theta = \dfrac{1}{\csc\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
$\csc\theta = \dfrac{1}{\sin\theta}$	$\sec\theta = \dfrac{1}{\cos\theta}$	$\cot\theta = \dfrac{1}{\tan\theta}$

Quotient Identities

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Sum and Difference Identities

$$\sin(a+b) = \sin a\cos b + \cos a\sin b$$

$$\sin(a-b) = \sin a\cos b - \cos a\sin b$$

$$\cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$\cos(a-b) = \cos a\cos b + \sin a\sin b$$

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a\tan b}$$

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a\tan b}$$

Half-Angle Identities

$$\cos\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1+\cos A}{2}}$$

$$\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{2}}$$

$$\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\sin A}{1+\cos A}$$

$$\tan\left(\frac{A}{2}\right) = \frac{1-\cos A}{\sin A}$$

Double-Angle Identities

$$\cos(2A) = \cos^2 A - \sin^2 A$$

$$\cos(2A) = 1 - 2\sin^2 A$$

$$\cos(2A) = 2\cos^2 A - 1$$

$$\sin(2A) = 2\sin A\cos A$$

$$\tan(2A) = \frac{2\tan A}{1 - \tan^2 A}$$

Let’s Get Started:

Watch the video below to see how to solve trig equations using identities.

Watch Video: Solving Trig Equations Using Identities

Tips

1. Do NOT divide by one of the trig functions. You will likely lose at least one solution.

2. Make sure the inputs in each trig function match. For instance sin(**x**) and sin(**2x**) have different inputs. You need to make them match. In this case change sin(2x) by using the double angle identity.

3. Make the trig functions match if possible. For instance if you have a cosine and a secant in your equation, rewrite the secant as 1/cos.

4. Notice when you have a quadratic. If one of your terms is squared, you are likely going to need to factor.

Solve $\cot x \cos^2 x = 2 \cot x$.

Example #1:

Solution

Begin by rewriting the equation so that all terms are collected on one side of the equation.

$$\cot x \cos^2 x = 2 \cot x \qquad \text{Write original equation.}$$

$$\cot x \cos^2 x - 2 \cot x = 0 \qquad \text{Subtract } 2 \cot x \text{ from each side.}$$

$$\cot x(\cos^2 x - 2) = 0 \qquad \text{Factor.}$$

By setting each of these factors equal to zero, you obtain

$$\cot x = 0 \quad \text{and} \quad \cos^2 x - 2 = 0$$

$$x = \frac{\pi}{2} \qquad \cos^2 x = 2$$

$$\cos x = \pm\sqrt{2}.$$

The equation $\cot x = 0$ has the solution $x = \pi/2$ [in the interval $(0, \pi)$]. No solution is obtained for $\cos x = \pm\sqrt{2}$ because $\pm\sqrt{2}$ are outside the range of the cosine function. Because $\cot x$ has a period of π, the general form of the solution is obtained by adding multiples of π to $x = \pi/2$, to get

$$x = \frac{\pi}{2} + n\pi \qquad \text{General solution}$$

where n is an integer. You can confirm this graphically by sketching the graph of $y = \cot x \cos^2 x - 2 \cot x$, as shown in Figure 5.8. From the graph you can see that the x-intercepts occur at $-3\pi/2$, $-\pi/2$, $\pi/2$, $3\pi/2$, and so on. These x-intercepts correspond to the solutions of $\cot x \cos^2 x - 2 \cot x = 0$.

Example #2: Solve $2\sin^2 x + 3\cos x - 3 = 0$.

Solution

This equation contains both sine and cosine functions. You can rewrite the equation so that it has only cosine functions by using the identity $\sin^2 x = 1 - \cos^2 x$.

$$\begin{aligned} 2\sin^2 x + 3\cos x - 3 &= 0 && \text{Write original equation.} \\ 2(1 - \cos^2 x) + 3\cos x - 3 &= 0 && \text{Pythagorean identity} \\ 2\cos^2 x - 3\cos x + 1 &= 0 && \text{Multiply each side by } -1. \\ (2\cos x - 1)(\cos x - 1) &= 0 && \text{Factor.} \end{aligned}$$

Set each factor equal to zero to find the solutions in the interval $[0, 2\pi)$.

$$2\cos x - 1 = 0 \quad \Longrightarrow \quad \cos x = \frac{1}{2} \quad \Longrightarrow \quad x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\cos x - 1 = 0 \quad \Longrightarrow \quad \cos x = 1 \quad \Longrightarrow \quad x = 0$$

Because $\cos x$ has a period of 2π, the general form of the solution is obtained by adding multiples of 2π to get

$$x = 2n\pi, \quad x = \frac{\pi}{3} + 2n\pi, \quad x = \frac{5\pi}{3} + 2n\pi \qquad \text{General solution}$$

where n is an integer.

Example #3:

Find all solutions of $\cos x + 1 = \sin x$ in the interval $[0, 2\pi)$.

Solution

It is not clear how to rewrite this equation in terms of a single trigonometric function. Notice what happens when you square each side of the equation.

$$\cos x + 1 = \sin x \qquad \text{Write original equation.}$$

$$\cos^2 x + 2\cos x + 1 = \sin^2 x \qquad \text{Square each side.}$$

$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x \qquad \text{Pythagorean identity}$$

$$\cos^2 x + \cos^2 x + 2\cos x + 1 - 1 = 0 \qquad \text{Rewrite equation.}$$

$$2\cos^2 x + 2\cos x = 0 \qquad \text{Combine like terms.}$$

$$2\cos x(\cos x + 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero produces

$$2\cos x = 0 \qquad \text{and} \qquad \cos x + 1 = 0$$

$$\cos x = 0 \qquad\qquad \cos x = -1$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad\qquad x = \pi.$$

Because you squared the original equation, check for extraneous solutions.

Check $x = \pi/2$

$$\cos\frac{\pi}{2} + 1 \stackrel{?}{=} \sin\frac{\pi}{2} \qquad \text{Substitute } \pi/2 \text{ for } x.$$

$$0 + 1 = 1 \qquad \text{Solution checks. } \checkmark$$

Check $x = 3\pi/2$

$$\cos\frac{3\pi}{2} + 1 \stackrel{?}{=} \sin\frac{3\pi}{2} \qquad \text{Substitute } 3\pi/2 \text{ for } x.$$

$$0 + 1 \neq -1 \qquad \text{Solution does not check.}$$

Check $x = \pi$

$$\cos\pi + 1 \stackrel{?}{=} \sin\pi \qquad \text{Substitute } \pi \text{ for } x.$$

$$-1 + 1 = 0 \qquad \text{Solution checks. } \checkmark$$

Example #4:

Solve $2\cos 3t - 1 = 0$.

Solution

$$2\cos 3t - 1 = 0 \qquad \text{Write original equation.}$$

$$2\cos 3t = 1 \qquad \text{Add 1 to each side.}$$

$$\cos 3t = \frac{1}{2} \qquad \text{Divide each side by 2.}$$

In the interval $[0, 2\pi)$, you know that $3t = \pi/3$ and $3t = 5\pi/3$ are the only solutions, so, in general, you have

$$3t = \frac{\pi}{3} + 2n\pi \qquad \text{and} \qquad 3t = \frac{5\pi}{3} + 2n\pi.$$

Dividing these results by 3, you obtain the general solution

$$t = \frac{\pi}{9} + \frac{2n\pi}{3} \qquad \text{and} \qquad t = \frac{5\pi}{9} + \frac{2n\pi}{3} \qquad \text{General solution}$$

where n is an integer.

Example #5: Solve $\sec^2 x - 2\tan x = 4$.

Solution

$$\sec^2 x - 2\tan x = 4 \qquad \text{Write original equation.}$$

$$1 + \tan^2 x - 2\tan x - 4 = 0 \qquad \text{Pythagorean identity}$$

$$\tan^2 x - 2\tan x - 3 = 0 \qquad \text{Combine like terms.}$$

$$(\tan x - 3)(\tan x + 1) = 0 \qquad \text{Factor.}$$

Setting each factor equal to zero, you obtain two solutions in the interval $(-\pi/2, \pi/2)$. [Recall that the range of the inverse tangent function is $(-\pi/2, \pi/2)$.]

$$\tan x - 3 = 0 \qquad \text{and} \qquad \tan x + 1 = 0$$

$$\tan x = 3 \qquad\qquad \tan x = -1$$

$$x = \arctan 3 \qquad\qquad x = -\frac{\pi}{4}$$

Finally, because $\tan x$ has a period of π, you obtain the general solution by adding multiples of π

$$x = \arctan 3 + n\pi \qquad \text{and} \qquad x = -\frac{\pi}{4} + n\pi \qquad \text{General solution}$$

where n is an integer. You can use a calculator to approximate the value of $\arctan 3$.

Example #6: Solve $2\cos x + \sin 2x = 0$.

Solution

Begin by rewriting the equation so that it involves functions of x (rather than $2x$). Then factor and solve.

$$2\cos x + \sin 2x = 0 \qquad \text{Write original equation.}$$

$$2\cos x + 2\sin x\cos x = 0 \qquad \text{Double-angle formula}$$

$$2\cos x(1 + \sin x) = 0 \qquad \text{Factor.}$$

$$2\cos x = 0 \quad \text{and} \quad 1 + \sin x = 0 \qquad \text{Set factors equal to zero.}$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2} \qquad x = \frac{3\pi}{2} \qquad \text{Solutions in } [0, 2\pi)$$

So, the general solution is

$$x = \frac{\pi}{2} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{2} + 2n\pi$$

where n is an integer. Try verifying these solutions graphically.

Practice

On a separate piece of paper, solve the following equations keeping in mind the identities and techniques covered in the last several lessons.

1. $2\cos x + 1 - \sin^2 x = 3$

2. $\cos 2x = \cos x$

3. $3\cos x + 2 = 5\sec x$

Practice - **ANSWERS**

On a separate piece of paper, solve the following equations keeping in mind the identities and techniques covered in the last several lessons.

Watch the following video for worked out solutions for each of the problems below.

Video: 6.4 Solving Trigonometric Equations Using Identities

Skip to the times below for solutions.

0:36 1. $2\cos x + 1 - \sin^2 x = 3$

3:13 2. $\cos 2x = \cos x$

6:57 3. $3\cos x + 2 = 5\sec x$

***** Please note a mistake on problem 3. It does not affect the final answer, however one of his factors should have been (3cos x + 5) NOT (3cos x - 5).**

Additional Resource Videos:

Solving Trigonometric Equations Using Identities and Substitution

Solving Trigonometric Equations Using Identities, Multiple Angles, By Factoring, General Solution

Additional Practice:

Trig Equations with Factoring & Fundamental Identities

The link above has a key, but the following link as worked out solutions: Answer Key - Worked Out Solutions

Equations and Multiple-Angle Identities

Problems 1-8